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Interfacial viscoelastic SH waves

Maurizio Romeo *

D.I.B.E. Università, via Opera Pia 11/a, 16145 Genova, Italy

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Abstract

Shear horizontal waves, in the form of transient perturbations, are considered at the interface between two different viscoelastic solids. The admissibility of these interfacial waves is studied via the asymptotic expansion of the Laplace transform of the viscoelastic kernel. The compatibility condition is reduced to a set of algebraic systems which can be solved iteratively to the desired order in the asymptotic expansion. Two classes of solutions are found which correspond to transient waves decaying away from the interface and attenuated along the propagation direction. Numerical examples are given to illustrate the results.

Keywords: Interface; Viscoelasticity; Shear waves

1. Introduction

It is known that the plane interface between two different elastic solids can support mechanical disturbances, propagating in some direction along the interface, whose amplitude rapidly decreases with the depth into the bulk of the two solids. Such perturbations, known as Stoneley waves, were originally discussed under the assumption of monochromatic stationary modes within the linear theory of isotropic elastic solids (see Stoneley, 1924; Sezawa and Kanai, 1939; Scholte, 1947). As shown in these works, Stoneley waves with a given polarization are admitted for some ranges of values of the elastic moduli and densities in the two half-spaces. This peculiarity may be relevant for technological applications of interfacial waves which, like Rayleigh waves, can be exploited for non-destructive testing of materials or for propagation of pulses in acoustic devices (see for e.g. Goudra and Stawiski, 2000; Hussain and Ogden, 2000).

Non-stationary interfacial waves have been rarely considered in literature even in the simple case of isotropic media. In this regard we mention the paper by Chadwick (1976) on in-plane Stoneley waves at the interface of isotropic elastic half-spaces.

Concerning anisotropic solids, the analysis of stationary interfacial waves produces more interesting results (see Barnett et al., 1985; Abbudi and Barnett, 1990; Barnett, 2000). For instance, unlike Rayleigh waves, for any set of material parameters and for a given orientation of the interface, forbidden directions

* Fax: +39-10-353-2777.
E-mail address: romeo@dibe.unige.it (M. Romeo).

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exist along which Stoneley waves cannot propagate. On the other hand, it should be remarked that, in contrast to surface waves, interfacial waves can be polarized as shear waves along the surface. In particular, if two different half-spaces admit a plane of material symmetry which is perpendicular to the plane interface, shear horizontal (SH) waves polarized along the normal to the plane of symmetry can propagate for suitable values of the material parameters. The aim of the present work is to analyze such modes in the more general context of transient waves. This approach allows us to discuss the occurrence of wave modes generated by a mechanical SH pulse of arbitrary shape, switched on at the interface and decaying away from it. Moreover, we account for dissipative effects due to the presence of viscoelasticity and inquire into the compatibility of interfacial waves with the thermodynamic restrictions on the viscoelastic tensor.

The constitutive model is outlined in Section 2 according to the well established theory of linear viscoelasticity. In Section 3 we derive the compatibility conditions for SH waves exploiting a separation of variables and requiring the continuity of the mechanical displacement and of the traction across the interface between the viscoelastic half-spaces. The compatibility condition for transient modes is written in terms of the Laplace transforms of the constitutive kernels and its asymptotic analysis is performed in Section 4. Here we show that, up to the desired order in the asymptotic expansion, the transformed problem can be reduced to a set of algebraic systems to be solved iteratively. Real and complex-valued solutions are discussed in Section 5 and the Laplace transform of the corresponding propagators are inverted. It turns out that two classes of solutions to the interface problem exist in the form of transient waves. In the first class the wave amplitude decays exponentially according to an attenuation factor due to viscoelasticity. In the second class the amplitude decreases as the reciprocal of the distance from the interface, multiplied by the viscoelastic attenuation factor. Numerical examples are given in Section 6 to illustrate the wave behavior due to a square pulse switched on at the interface.

2. Viscoelastic interfaces

Let $\mathscr{S}$ be a plane interface between two viscoelastic anisotropic and homogeneous half-spaces $\mathscr{B}_1$ and $\mathscr{B}_2$. We assume that an axis $\mathbf{e}$ exists along S which is a common twofold axis of symmetry or the normal to a common plane of symmetry for both $\mathscr{B}_1$ and $\mathscr{B}_2$. We choose Cartesian orthogonal axes $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ with $\mathbf{e}_z = \mathbf{e}$ and such that $\mathbf{e}_y$ be normal to $\mathscr{S}$ and directed towards the interior of $\mathscr{B}_1$ (see Fig. 1). According to the linear theory of viscoelasticity we write the Cauchy stress $\mathbf{T}(\mathbf{x}, t)$ as a linear functional of the history of the infinitesimal strain $\mathbf{E}(\mathbf{x}, t) = \frac{1}{2}[\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}}](\mathbf{x}, t)$, where $\mathbf{u}(\mathbf{x}, t)$ is the mechanical displacement. Hence the following constitutive equation is assumed for both the regions $\mathscr{B}_1$ and $\mathscr{B}_2$,

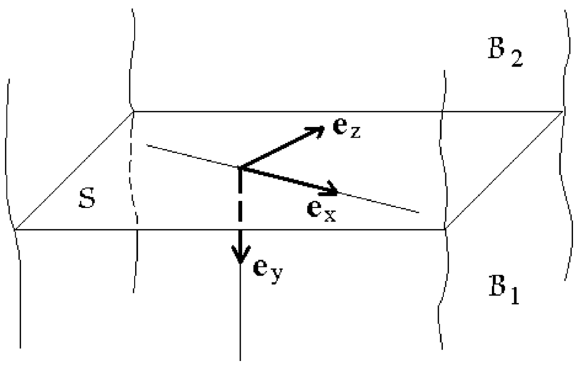

Fig. 1. Geometry of the SH wave problem at an interface.

$$\mathbf{T}(\mathbf{x},t) = \mathbf{G}_0\mathbf{E}(\mathbf{x},t) + \int_0^\infty \mathbf{G}'(\tau)\mathbf{E}(\mathbf{x},t-\tau)\,\mathrm{d}\tau, \tag{2.1}$$

where $\mathbf{G}(t)$ is the viscoelastic (fourth-order) symmetric tensor and $\mathbf{G}_0 = \mathbf{G}(0)$ represents the instantaneous elastic modulus. In the following we shall take $\mathbf{G} \in C^\infty(\mathbb{R}^+)$ and, as usual, the tensor $\mathbf{G}_0$ will be assumed positive definite. In addition, as a consequence of the second law of thermodynamics, $\mathbf{G}'(t)$ turns out to be negative semidefinite at $t = 0$ (see Fabrizio and Morro, 1988). Adopting the customary six-dimensional notation, the previous hypotheses on the material symmetry of $\mathscr{B}_1$ and $\mathscr{B}_2$ amount to the following restrictions on the entries of $\mathbf{G}$,

$$G_{\alpha\beta}(t) = G_{\beta\alpha}(t) = 0 \quad \text{for} \quad \alpha = 4,5, \;\; \beta \neq 4,5, \;\; t \in \mathbb{R}^+, \tag{2.2}$$

while the thermodynamic inequalities imply, in particular,

$$\begin{aligned} &G^0_{44} > 0, \quad G^0_{55} > 0, \quad G^0_{44}G^0_{55} - {G^0_{45}}^2 > 0, \\ &G'_{44} \leqslant 0, \quad G'_{55} \leqslant 0, \quad G'_{44}G'_{55} - {G'_{45}}^2 \geqslant 0 \quad \text{at} \quad t = 0, \end{aligned} \tag{2.3}$$

where $G^0_{\alpha\beta} = G_{\alpha\beta}(0)$.

Obviously, the two half-spaces $\mathscr{B}_1$ and $\mathscr{B}_2$ comply with different values of the entries $G^0_{\alpha\beta}$ and $G'_{\alpha\beta}(0)$. The cases of two purely elastic half-spaces or of an elastic half-space, say $\mathscr{B}_1$, matched with a viscoelastic $\mathscr{B}_2$ will be characterized respectively by $G'_{\alpha\beta} = 0$ identically in both $\mathscr{B}_1$ and $\mathscr{B}_2$ or only in $\mathscr{B}_1$. Special transient modes in these cases will be briefly discussed in the last section.

3. Compatibility conditions for SH waves

In the following we deal with waves propagating along the sagittal plane spanned by $\mathbf{e}_x$ and $\mathbf{e}_y$, within $\mathscr{B}_1$ and $\mathscr{B}_2$. As a consequence we suppose that the perturbation is independent on z. Then, according to Eqs. (2.1) and (2.2), the equations of motion take the following form:

$$\begin{aligned} \rho u_{x,tt} &= (\mathscr{G}_{11}[u_x] + \mathscr{G}_{16}[u_y])_{,xx} + (\mathscr{G}_{66}[u_x] + \mathscr{G}_{26}[u_y])_{,yy} + \{2\mathscr{G}_{16}[u_x] + (\mathscr{G}_{12} + \mathscr{G}_{66})[u_y]\}_{,xy}, \\ \rho u_{y,tt} &= (\mathscr{G}_{16}[u_x] + \mathscr{G}_{66}[u_y])_{,xx} + (\mathscr{G}_{26}[u_x] + \mathscr{G}_{22}[u_y])_{,yy} + \{2\mathscr{G}_{26}[u_y] + (\mathscr{G}_{12} + \mathscr{G}_{66})[u_x]\}_{,xy}, \end{aligned} \tag{3.1}$$

$$\rho u_{z,tt} = \mathscr{G}_{55}[u_{z,xx}] + \mathscr{G}_{44}[u_{z,yy}] + 2\mathscr{G}_{45}[u_{z,xy}], \tag{3.2}$$

where commas denote partial differentiation and where $\mathscr{G}_{\alpha\beta}$ are integral operators defined as

$$\mathscr{G}_{\alpha\beta}[f](t) = G^0_{\alpha\beta}f(t) + \int_0^t G'_{\alpha\beta}(\tau)f(t-\tau)\,\mathrm{d}\tau, \quad \alpha,\beta = 1,\ldots,6$$

for any field $f \in L^1(\mathbb{R}^+)$. In doing so we have implicitly assumed that perturbations are absent for $t < 0$. We also require the continuity of the mechanical displacement $\mathbf{u}$ and of the traction $\mathbf{T}\mathbf{e}_y$ across $\mathscr{S}$. By choosing the origin of the coordinate system at the interface $\mathscr{S}$, we get

$$[\![\mathscr{G}_{\alpha 1}[u_{x,x}] + \mathscr{G}_{\alpha 6}[u_{y,x} + u_{x,y}] + \mathscr{G}_{\alpha 2}[u_{y,y}]]\!] = 0, \quad \alpha = 2,6, \;\; [\![u_x]\!] = [\![u_y]\!] = 0, \tag{3.3}$$

$$[\![\mathscr{G}_{45}[u_{z,x}] + \mathscr{G}_{44}[u_{z,y}]]\!] = 0, \quad [\![u_z]\!] = 0, \tag{3.4}$$

where $[\![f]\!] = f|_{y=0^+} - f|_{y=0^-}$. In the stationary case, interfacial waves are defined as perturbations with non-vanishing amplitudes in the neighborhood of $\mathscr{S}$ and an additional asymptotic condition is required to allow for their damping at large distances from $\mathscr{S}$. Considering transient solutions to the dynamic problem, it will be sufficient to impose that $\mathbf{u}$ be bounded for any $y \in \mathbb{R}$.

In view of Eqs. (3.1)–(3.4), the problem decouples into two distinct systems concerning waves polarized on the sagittal plane and shear horizontal (SH) waves polarized on $\mathbf{e}_z$. In the following we shall be interested only in SH solutions governed by Eqs. (3.2), (3.4) and pose $u_z = u$.

In order to study the occurrence of transient waves propagating along the surface $\mathscr{S}$, we adopt a separation of dependence on the space variables x and y, by writing the displacement u in the form of a convolution of split fields,

$$u(x,y,t) = [u_a(x,\cdot) * u_b(y,\cdot)](t) \tag{3.5}$$

where $(a*b)(t) = \int_0^t a(\tau)b(t-\tau)\,\mathrm{d}\tau$. Accordingly, denoting by $\hat{f}(s)$ the Laplace transform of $f(t)$, from Eqs. (3.2) and (3.4) we obtain the following transformed problem

$$\rho s^2 \hat{u}_a \hat{u}_b = s\hat{\mathscr{G}}_{44}\hat{u}_a\hat{u}_{b,yy} + s\hat{\mathscr{G}}_{55}\hat{u}_b\hat{u}_{a,xx} + 2s\hat{\mathscr{G}}_{45}\hat{u}_{a,x}\hat{u}_{b,y}, \tag{3.6}$$

$$[\![\hat{\mathscr{G}}_{45}\hat{u}_{a,x}\hat{u}_b + \hat{\mathscr{G}}_{44}\hat{u}_a\hat{u}_{b,y}]\!] = 0, \quad [\![\hat{u}_b]\!] = 0. \tag{3.7}$$

Solutions to Eq. (3.6) can be written in the form

$$\begin{aligned} \hat{u}_a(x,s) &= a(s)\exp[K(s)x], \\ \hat{u}_b(y,s) &= b(s)\exp[H(s)y]. \end{aligned} \tag{3.8}$$

With the requirement of boundedness in $\mathscr{B}_1$ and $\mathscr{B}_2$, we must have

$$\hat{u}_b(y,s) = \begin{cases} b_1(s)\exp[H_1(s)y], & \text{for } y > 0, \\ b_2(s)\exp[-H_2(s)y], & \text{for } y < 0, \end{cases} \tag{3.9}$$

$$\Re H_1(s) \leqslant 0, \quad \Re H_2(s) \leqslant 0, \quad \forall s \in \mathbb{C}. \tag{3.10}$$

In view of Eqs. (3.8), the problems (3.6) and (3.7) take the form

$$\begin{aligned} \rho_1 s^2 &= s\hat{\mathscr{G}}_{44}^{(1)}(s)H_1^2(s) + s\hat{\mathscr{G}}_{55}^{(1)}(s)K^2(s) + 2s\hat{\mathscr{G}}_{45}^{(1)}(s)K(s)H_1(s), \\ \rho_2 s^2 &= s\hat{\mathscr{G}}_{44}^{(2)}(s)H_2^2(s) + s\hat{\mathscr{G}}_{55}^{(2)}(s)K^2(s) - 2s\hat{\mathscr{G}}_{45}^{(2)}(s)K(s)H_2(s), \end{aligned} \tag{3.11}$$

$$\begin{aligned} &\hat{\mathscr{G}}_{45}^{(1)}(s)K(s) + \hat{\mathscr{G}}_{44}^{(1)}(s)H_1(s) = \hat{\mathscr{G}}_{45}^{(2)}(s)K(s) - \hat{\mathscr{G}}_{44}^{(2)}(s)H_2(s), \\ &b_1(s) = b_2(s) =: b(s), \end{aligned} \tag{3.12}$$

where the superscripts (1) and (2) denote quantities pertaining to $\mathscr{B}_1$ and $\mathscr{B}_2$ respectively. Eqs. (3.11) and (3.12), together with the conditions (3.10) represent the compatibility conditions for SH waves propagating along the interface $\mathscr{S}$.

4. Asymptotic analysis of the compatibility conditions

Here we assume that the viscoelastic tensor can be expanded in Taylor series in the neighborhood of $t = 0$. Then, by the Watson's lemma, we obtain the following expansions for the Laplace transforms of the quantities $\mathscr{G}_{\alpha\beta}$,

$$
\begin{aligned}
s\hat{\mathscr{G}}_{44}^{(q)}(s) &= A_0^{(q)} + \sum_{k=1}^{N} \frac{A_k^{(q)}}{s^k} + R_{Aq}^N(s),\\
s\hat{\mathscr{G}}_{55}^{(q)}(s) &= C_0^{(q)} + \sum_{k=1}^{N} \frac{C_k^{(q)}}{s^k} + R_{Cq}^N(s), \quad q = 1,2,\\
s\hat{\mathscr{G}}_{45}^{(q)}(s) &= B_0^{(q)} + \sum_{k=1}^{N} \frac{B_k^{(q)}}{s^k} + R_{Bq}^N(s),
\end{aligned}
\tag{4.1}
$$

where, according to the pertinent half-space, $A_0 = G_{44}^0$, $C_0 = G_{55}^0$, $B_0 = G_{45}^0$, and where A_k, C_k, and B_k are, respectively, the kth derivatives of G_{44}, G_{55} and G_{45}, valued at $t = 0$. The functions $R^N(s)$ in the right hand sides of (4.1) are suitable remainder terms satisfying the requirements

$$
|R^N(s)| \leqslant M_{N,\xi}|s|^{-N-1} \quad \text{for} \quad \Re s \geqslant a, \;\; |\arg s| \leqslant \frac{\pi}{2} - \xi \tag{4.2}
$$

with $a, \xi \in \mathbb{R}^{++}$ and $M_{N,\xi} \in \mathbb{R}^{+}$. If Eqs. (4.1) are substituted into (3.11) we realize that $H_1(s)$, $H_2(s)$ and $K(s)$ must have a pole of order one at $s = \infty$ and the following expansions hold:

$$
\begin{aligned}
K(s) &= k_0 s + k_1 + \sum_{m=1}^{N} \frac{k_{m+1}}{s^m} + R_K^N(s),\\
H_1(s) &= h_0^{(1)} s + h_1^{(1)} + \sum_{m=1}^{N} \frac{h_{m+1}^{(1)}}{s^m} + R_1^N(s),\\
H_2(s) &= h_0^{(2)} s + h_1^{(2)} + \sum_{m=1}^{N} \frac{h_{m+1}^{(2)}}{s^m} + R_2^N(s),
\end{aligned}
\tag{4.3}
$$

where R_K^N, R_1^N, R_2^N satisfy conditions analogous to (4.2). Substitution of Eqs. (4.3) and (4.1) into Eqs. (3.11) and (3.12) yields a system for the expansion coefficients k_m, $h_m^{(1)}$, $h_m^{(2)}$, $(m = 0, 1, 2, \ldots, N+1)$. Disregarding the remainder terms R^N and equating terms of the same order in s, this system turns out to be equivalent to a set of $N+2$ systems which can be solved iteratively starting from the following zeroth order equations:

$$
\begin{aligned}
\rho_1 &= A_0^{(1)} h_0^{(1)^2} + C_0^{(1)} k_0^2 + 2B_0^{(1)} h_0^{(1)} k_0,\\
\rho_2 &= A_0^{(2)} h_0^{(2)^2} + C_0^{(2)} k_0^2 - 2B_0^{(2)} h_0^{(1)} k_0,\\
B_0^{(1)} k_0 + A_0^{(1)} h_0^{(1)} &= B_0^{(2)} k_0 - A_0^{(2)} h_0^{(2)}.
\end{aligned}
\tag{4.4}
$$

We remark that, if $B_0^{(1)} = B_0^{(2)}$, Eq. $(4.4)_3$ implies $A_0^{(1)} h_0^{(1)} + A_0^{(2)} h_0^{(2)} = 0$. In view of inequalities (2.3) and (3.10), this equation does not admit solutions in the form of interfacial waves. In particular, it applies if $\mathscr{B}_1$ and $\mathscr{B}_2$ are such that $B_0^{(1)} = B_0^{(2)} = 0$. In this case the previous solution corresponds to a transient bold transverse wave occurring if $C_0^{(1)}/\rho_1 = C_0^{(2)}/\rho_2 = c_T^2$, where c_T is the common speed of SH bold waves propagating along the x-axis in $\mathscr{B}_1$ and $\mathscr{B}_2$. The counterpart of this solution in the stationary isotropic case is depicted in the case 2 in Barnett et al. (1985, Section 4).

In the general case, when $B_0^{(1)} \neq B_0^{(2)}$, accounting for inequalities (3.10), Eqs. (4.4) admit the following solutions:

$$
k_0^2 = \frac{A_0^{(2)} \rho_2 - A_0^{(1)} \rho_1}{\alpha_2 - \alpha_1}, \quad h_{0\pm}^{(q)} = \frac{(-1)^q}{A_0^{(q)}} \left[-\frac{B_0^{(q)}}{c_k} \pm \beta \right], \quad q = 1,2 \tag{4.5}
$$

with

$$\beta = \sqrt{\frac{A_0^{(1)}\rho_1\alpha_2 - A_0^{(2)}\rho_2\alpha_1}{\alpha_2 - \alpha_1}}, \tag{4.6}$$

and where $\alpha_1 = A_0^{(1)}C_0^{(1)} - B_0^{(1)^2}$, $\alpha_2 = A_0^{(2)}C_0^{(2)} - B_0^{(2)^2}$. Provided that

$$B_0^{(2)} > B_0^{(1)}, \quad (A_0^{(2)}\rho_2 - A_0^{(1)}\rho_1)(\alpha_2 - \alpha_1) > 0, \tag{4.7}$$

and without loss of generality, we can choose $\Re k_0 < 0$, which implies forward propagation along the x-axis, and write

$$k_0 = -\frac{1}{c_k}, \quad c_k = \sqrt{\frac{\alpha_2 - \alpha_1}{A_0^{(2)}\rho_2 - A_0^{(1)}\rho_1}}. \tag{4.8}$$

In view of Eq. (4.6), real solutions for $h_{0\pm}^{(q)}$ are obtained if the additional condition holds

$$(A_0^{(1)}\rho_1\alpha_2 - A_0^{(2)}\rho_2\alpha_1)(\alpha_2 - \alpha_1) \geqslant 0, \tag{4.9}$$

otherwise, complex conjugate solutions, compatible with (3.10), exist for $B_0^{(1)} < 0$, $B_0^{(2)} > 0$.

In absence of viscoelastic effects, the expansion coefficients k_m, $h_m^{(1)}$, $h_m^{(2)}$, $(m = 1, 2, \ldots)$ vanish identically and Eqs. (4.5)–(4.8) characterize all possible solutions for the problem at hand. As an illustrative example, elastic parameters which allow for such solutions are shown in Fig. 2 assuming that $\mathscr{B}_1$ consists of a crystal

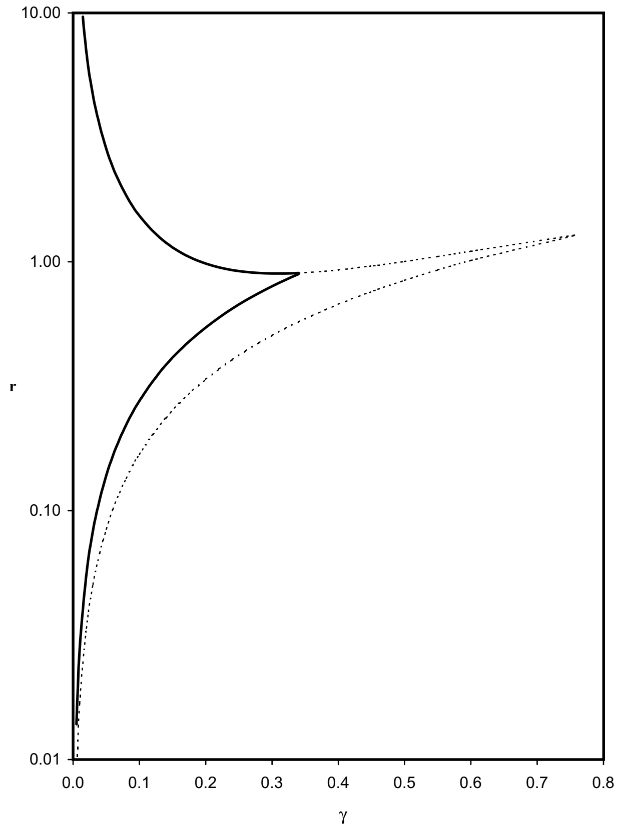

Fig. 2. Compatibility regions for elastic (dashed) and viscoelastic (solid) interfacial waves: $r = \rho_2/\rho_1$ and $\gamma = G_{44}^{(2)}/G_{44}^{(1)}$.

Table 1
Material parameters for the half-space $\mathscr{B}_1$ (quartz)

$\rho_1 = 2.65 \times 10^3$ kg/m^3
$G_{44}^{(1)} = 5.79 \times 10^{10}$ Pa
$G_{55}^{(1)} = 3.99 \times 10^{10}$ Pa
$G_{45}^{(1)} = -1.79 \times 10^{10}$ Pa

of quartz with normal to the plane of material symmetry directed as $\mathbf{e}_z$. The pertinent material data are given in Table 1. For definiteness we suppose that $\mathscr{B}_2$ belongs to a more symmetric crystal class such that $G_{44}^{(2)} = G_{55}^{(2)}$ and $G_{45}^{(2)} = 0$. In this instance, for not small values of the rate $\gamma = G_{44}^{(2)}/G_{44}^{(1)}$, compatible elastic solutions are obtained at density rates $r = \rho_2/\rho_1$ sufficiently close to 1.

Viscoelastic solutions are obtained by solving iteratively the following systems:

$$\begin{aligned}
&(A_0^{(1)}h_0^{(1)} + B_0^{(1)}k_0)h_m^{(1)} + (C_0^{(1)}k_0 + B_0^{(1)}h_0^{(1)})k_m = -\frac{1}{2}\widetilde{\sum_{i+j+l=m}} \mathbf{E}_i^{(1)} \cdot \mathbf{Q}_j^{(1)}\mathbf{E}_l^{(1)}, \\
&(A_0^{(2)}h_0^{(2)} - B_0^{(2)}k_0)h_m^{(2)} + (C_0^{(2)}k_0 - B_0^{(2)}h_0^{(2)})k_m = -\frac{1}{2}\widetilde{\sum_{i+j+l=m}} \mathbf{E}_i^{(2)} \cdot \mathbf{Q}_j^{(2)}\mathbf{E}_l^{(2)}, \\
&A_0^{(1)}h_m^{(1)} + A_0^{(2)}h_m^{(2)} + \left(B_0^{(1)} - B_0^{(2)}\right)k_m = -\sum_{i+j=m-1}\left[\left(B_i^{(1)} - B_i^{(2)}\right)k_j + A_i^{(1)}h_j^{(1)} + A_i^{(2)}h_j^{(2)}\right]
\end{aligned} \tag{4.10}$$

for $m = 1, 2, \ldots, N+1$, where

$$\mathbf{E}_i^{(q)} = \begin{pmatrix} h_i^{(q)} \\ k_i \end{pmatrix}, \quad \mathbf{Q}_j^{(q)} = \begin{pmatrix} A_j^{(q)} & (-1)^{q-1}B_j^{(q)} \\ (-1)^{q-1}B_j^{(q)} & C_j^{(q)} \end{pmatrix}, \quad q = 1, 2,$$

and where a superimposed tilde denotes summation with $i, l = 0, 1, \ldots, m-1$ and $j = 0, 1, \ldots, m$. Once the quantities $h_i^{(1)}$, $h_i^{(2)}$, k_i, $(i = 1, \ldots, m-1)$ are substituted into the right hand side of (4.10), a linear system is obtained for $h_m^{(1)}$, $h_m^{(2)}$ and k_m. It is easy to show that if $\{(h_m^{(1)}, h_m^{(2)}, k_m), m = 1, \ldots, N+1\}$ is a solution of Eqs. (4.10), then also its complex conjugate $\{(\bar{h}_m^{(1)}, \bar{h}_m^{(2)}, \bar{k}_m), m = 1, \ldots, N+1\}$ is a solution. In particular, for $m = 1$, using Eqs. (4.8), the solution of (4.10) can be written as

$$\begin{aligned}
h_{1\pm}^{(1)} = \pm\frac{1}{A_0^{(1)}\Omega\beta}\Bigg\{&\left[\frac{A_0^{(1)}}{c_k^2}Q_1^{(2)}[A_0^{(2)}, B_0^{(2)}] - \beta^2\left(A_1^{(2)}A_0^{(1)} - A_1^{(1)}A_0^{(2)}\right)\right]\left(\frac{\alpha_1}{c_k} \pm \beta B_0^{(1)}\right) \\
&- \frac{A_0^{(2)}}{c_k^2}Q_1^{(1)}\left[A_0^{(1)}, B_0^{(1)}\right]\left(\frac{\alpha_2}{c_k} \pm \beta B_0^{(1)}\right) \mp \frac{A_0^{(2)}}{c_k}\beta(\alpha_1 - \alpha_2)\left[\frac{2}{c_k}\left(A_1^{(1)}B_0^{(1)} - B_1^{(1)}A_0^{(1)}\right) \mp \beta A_1^{(1)}\right]\Bigg\},
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
h_{1\pm}^{(2)} = \pm\frac{1}{A_0^{(2)}\Omega\beta}\Bigg\{&\left[\frac{A_0^{(2)}}{c_k^2}Q_1^{(1)}[A_0^{(1)}, B_0^{(1)}] - \beta^2\left(A_1^{(1)}A_0^{(2)} - A_1^{(2)}A_0^{(1)}\right)\right]\left(\frac{\alpha_2}{c_k} \mp \beta B_0^{(2)}\right) \\
&- \frac{A_0^{(1)}}{c_k^2}Q_1^{(2)}\left[A_0^{(2)}, B_0^{(2)}\right]\left(\frac{\alpha_1}{c_k} \pm \beta B_0^{(2)}\right) \mp \frac{A_0^{(1)}}{c_k}\beta(\alpha_2 - \alpha_1)\left[\frac{2}{c_k}\left(B_1^{(2)}A_0^{(2)} - A_1^{(2)}B_0^{(2)}\right) \mp \beta A_1^{(2)}\right]\Bigg\},
\end{aligned} \tag{4.12}$$

$$k_1 = \frac{1}{\Omega}\left[\left(A_0^{(1)}Q_1^{(2)}\left[A_0^{(2)}, B_0^{(2)}\right] - A_0^{(2)}Q_1^{(1)}\left[A_0^{(1)}, B_0^{(1)}\right]\right)\frac{1}{c_k} + \left(A_1^{(2)}A_0^{(1)} - A_1^{(1)}A_0^{(2)}\right)\beta^2\right], \tag{4.13}$$

where $\Omega = (2/c_k)A_0^{(1)}A_0^{(2)}(\alpha_1 - \alpha_2)$, and where

$$Q_1^{(q)}[a,b] = A_1^{(q)}a^2 + C_1^{(q)}b^2 + 2(-1)^{q-1}B_1^{(q)}ab, \quad q = 1,2.$$

The signs in Eqs. (4.11) and (4.12) must be chosen according to the choice adopted in Eq. $(4.5)_2$. From Eq. (4.13) we realize that k_1 is real, while $h_{1\pm}^{(1)}$, $h_{1\pm}^{(2)}$ turn out to be real if the inequality (4.9) holds. Otherwise, they consist in pairs of complex conjugate solutions. In view of inequalities (3.10), viscoelastic solutions require the additional restrictions $\Re h_1^{(1)} \leqslant 0$, $\Re h_1^{(2)} \leqslant 0$. Moreover, the assumption of forward propagation along x, implies $k_1 \leqslant 0$. Elastic parameters which allow for these conditions are shown in Fig. 2 under the definite assumptions $A_1^{(q)} = -\kappa A_0^{(q)}$, $C_1^{(q)} = -\kappa C_0^{(q)}$, $B_1^{(q)} = \kappa B_0^{(q)}$, $(q = 1,2)$, $\kappa > 0$. It is worth remarking that SH interfacial waves compatible with a pair of elastic half-spaces may not be supported at interfaces between viscoelastic media characterized by the same values of $A_0^{(q)}$, $B_0^{(q)}$, $C_0^{(q)}$. In addition, in view of the present assumptions on $A_1^{(q)}$, $B_1^{(q)}$, $C_1^{(q)}$, the restrictions on the existence of viscoelastic interfacial waves turn out to be independent on the parameter κ which measures the extent of the dissipative effects.

5. Transient modes

We assume that a mechanical perturbation $u_0(t) = u(0,0,t)$ be given at the line $x = 0$, $y = 0$ on $\mathscr{S}$ for $t \geqslant 0$ and introduce the wave propagators $\mathscr{P}^{(1)}$ and $\mathscr{P}^{(2)}$, pertinent respectively to the half-spaces $\mathscr{B}_1$ and $\mathscr{B}_2$, via the following convolutions:

$$u^{(q)}(x,y,t) = \left[\mathscr{P}^{(q)}(x,y,\cdot) * u_0(\cdot)\right](t), \quad q = 1,2. \tag{5.1}$$

Performing the Laplace transformation of Eq. (3.5), taking into account Eqs. (3.8), (3.9) and (5.1), we get $a(s)b(s) = \hat{u}_0(s)$ and, up to terms of order N in the expansions (4.3),

$$\hat{\mathscr{P}}^{(q)}(x,y,s) = \exp\left[\tau_0^{(q)}(x,y)s + \tau_1^{(q)}(x,y) + \sum_{m=1}^{N} \frac{\tau_{m+1}^{(q)}(x,y)}{s^m}\right]. \tag{5.2}$$

where, adopting the notation $h_m^{(q)} = \zeta_m^{(q)} + \mathrm{i}\eta_m^{(q)}$ $(m = 0,1,\dots,N)$, $k_{m+1} = \xi_{m+1} + \mathrm{i}v_{m+1}$ $(m = 1,\dots,N)$,

$$\begin{aligned}
\tau_0^{(q)}(x,y) &= -\frac{x}{c_k} - (-1)^q\left(\zeta_0^{(q)} + \mathrm{i}\eta_0^{(q)}\right)y, \\
\tau_1^{(q)}(x,y) &= k_1 x - (-1)^q\left(\zeta_1^{(q)} + \mathrm{i}\eta_1^{(q)}\right)y, \\
\tau_{m+1}^{(q)} &= (\xi_{m+1} + \mathrm{i}v_{m+1})x - (-1)^q\left(\zeta_{m+1}^{(q)} + \mathrm{i}\eta_{m+1}^{(q)}\right)y, \quad m = 1,\dots,N.
\end{aligned} \tag{5.3}$$

If inequality (4.9) is satisfied, systems (4.4) and (4.10) admit real solutions and the transforms (5.2) can be inverted to obtain $\mathscr{P}^{(q)} = \mathscr{P}_a^{(q)}$, with

$$\begin{aligned}
\mathscr{P}_a^{(q)}(x,y,t) = \exp\left[k_1 x - (-1)^q\zeta_1^{(q)}y\right]&\Bigg\{\delta\left(t - \frac{x}{c_k} - (-1)^q\zeta_0^{(q)}y\right) \\
&+ \prod_{m=1}^{N}{}^{*} S_m\left(\xi_{m+1}x - (-1)^q\zeta_{m+1}^{(q)}y, t - \frac{x}{c_k} - (-1)^q\zeta_0^{(q)}y\right)\Bigg\}, \quad q = 1,2,
\end{aligned} \tag{5.4}$$

where

$$S_m(\sigma,\tau) = -\frac{\tau^{m-1}}{(m-1)!}\sigma\, {}_0F_m\left(\frac{m+1}{2}, \frac{m+2}{2}, \dots, 2; -\left(\frac{\tau}{m}\right)^m\sigma\right), \quad m = 1,\dots,N.$$

The quantities ${}_0F_m$ represent hypergeometric functions and the symbol $\prod_{m=1}^{N} *$ stands for the composition of N convolutions on $(0,t)$. In view of Eq. (5.4), transient SH interfacial waves propagate along the surface $\mathscr{S}$

with speed c_k and their amplitude is attenuated in $\mathscr{B}_1$ and $\mathscr{B}_2$ according to the damping factors $\exp[k_1 x - (-1)^q \zeta_1^{(q)} y]$, $(q = 1, 2)$. We observe that the wave fronts $t = (x/c_k) + (-1)^q \zeta_0^{(q)} y$, $(q = 1, 2)$, propagate with the speeds

$$c^{(q)} = \frac{c_k}{\sqrt{1 + c_k^2 (\zeta_0^{(q)})^2}}, \quad q = 1, 2, \tag{5.5}$$

and, in general, are not parallel to the respective planes of constant amplitudes $k_1 x - (-1)^{(q)} \zeta_1^{(q)} y = \text{const}$. In this respect, interfacial SH waves governed by the propagator (5.4), consist of inhomogeneous waves (see for e.g. Caviglia and Morro, 1992).

If inequality (4.9) is not satisfied, $\tau_0^{(q)}$, $\tau_1^{(q)}$, $\tau_{m+1}^{(q)}$ $(m = 1, \ldots, N)$ take complex values in the form of conjugate pairs. In this case it is also possible to obtain transient solutions which satisfy inequalities (3.10). In fact, exploiting the existence of complex conjugate solutions for $h_m^{(q)}$, we can write Eq. (5.2) in the form

$$\hat{\mathscr{P}}^{(q)}(x, y, s) = \hat{\mathscr{P}}_a^{(q)}(x, y, s) \hat{\mathscr{P}}_b^{(q)}(x, y, s), \quad q = 1, 2, \tag{5.6}$$

where $\hat{\mathscr{P}}_a^{(q)}$ is the Laplace transform of (5.4) and

$$\hat{\mathscr{P}}_b^{(q)}(x, y, s) = \begin{cases} \exp\left[\mathrm{i}(-1)^{q-1} U(s) x + \mathrm{i}\left(\eta_1^{(q)} + V^{(q)}(s)\right) y\right], & \Im V^{(q)}(s) \geqslant 0, \\ \exp\left[-\mathrm{i}(-1)^{q-1} U(s) x - \mathrm{i}\left(\eta_1^{(q)} + V^{(q)}(s)\right) y\right], & \Im V^{(q)}(s) < 0, \end{cases} \quad q = 1, 2, \tag{5.7}$$

$$U(s) = \sum_{m=1}^{N} \frac{v_{m+1}}{s^m}, \quad V^{(q)}(s) = \eta_0^{(q)} s + \sum_{m=1}^{N} \frac{\eta_{m+1}^{(q)}}{s^m}, \quad q = 1, 2.$$

In the last expression, without loss of generality we have supposed $\eta_0^{(q)} > 0$ $(q = 1, 2)$. The transforms (5.7) are similar to those obtained in the study of surface SH transient waves in piezoelectric media (see Romeo, 2001). According to the procedure adopted in that context, the Laplace inverse transform of (5.7) can be written as

$$\mathscr{P}_b^{(q)}(x, y, t) = \frac{1}{\pi \eta_0^{(q)}} \int_0^{t/\eta_0^{(q)}} \left[\delta\left(t - \eta_0^{(q)} \omega\right) + \prod_{m=1}^{N}{}^{*} S_m\left((-1)^{q-1} v_{m+1} \frac{x}{y} + \eta_{m+1}^{(q)}, t - \eta_0^{(q)} \omega\right)\right] \times \frac{y \cos(\eta_1^{(q)} y) - \omega \sin(\eta_1^{(q)} y)}{y^2 + \omega^2} \, \mathrm{d}\omega, \quad q = 1, 2. \tag{5.8}$$

The wave propagators $\mathscr{P}^{(q)}$ in this case are given by the convolution of Eqs. (5.4) and (5.8). The resulting transient modes are inhomogeneous and characterized by wave speeds and attenuation factor in the same form of the previous case. Qualitative changes pertain mainly to the profile of the wave amplitude.

6. Examples

In order to illustrate the results obtained in the previous section we consider here two examples of viscoelastic interfaces where constitutive parameters satisfy inequalities (4.7) and allow for viscoelastic wave modes. In both instances we shall assume that the half-space $\mathscr{B}_1$ be determined by the parameters in Table 1.

In the first example we suppose, according to the setting of Fig. 2, that $G_{55}^{(2)} = G_{44}^{(2)}$, $G_{45}^{(2)} = 0$. Among the values of the parameters which turn out to be consistent with the existence of viscoelastic interfacial waves we consider $\gamma = 0.05$ and $r = 0.38$ which correspond to an half-space $\mathscr{B}_2$ made of ice. Unfortunately, definite data on the viscoelastic quantities $A_1^{(q)}$, $B_1^{(q)}$, $C_1^{(q)}$ and in turn, on the parameter κ are not available. In the following we take $\kappa = 0.1$. With these choices inequality (4.9) turns out to be satisfied and real

Table 2
Wave parameters for the first example

$k_0 = -2.756 \times 10^{-4}$ s/m	$k_1 = -2.269 \times 10^{-5}$ m^{-1}	$c_k = 3628$ m/s
$h_0^{(1)} = -5.914 \times 10^{-5}$ s/m	$h_1^{(1)} = -2.299 \times 10^{-5}$ m^{-1}	$c^{(1)} = 3547$ m/s
$h_0^{(2)} = -5.214 \times 10^{-4}$ s/m	$h_1^{(2)} = -2.136 \times 10^{-5}$ m^{-1}	$c^{(2)} = 1696$ m/s

admissible values of k_0, k_1, $h_0^{(q)}$, $h_1^{(q)}$, $q = 1, 2$, are obtained from (4.5) and (4.11)–(4.13). These results, together with the corresponding wave speeds c_k, $c^{(1)}$, $c^{(2)}$, given by Eqs. (4.8) and (5.5), are shown in Table 2. Now we suppose that a square pulse of width $\tau_{\rm p}$ is switched on at $x = 0$, $y = 0$ for $t = 0$, i.e.,

$$u_0(t) = \mathscr{H}(t)\mathscr{H}(\tau_{\rm p} - t) \tag{6.1}$$

where $\mathscr{H}$ is the Heaviside's unit step function. The propagating field is obtained by the convolution $(\mathscr{P}_a^{(q)} * u_0)(t)$ $(q = 1, 2)$, where $\mathscr{P}_a^{(q)}$ are given by Eq. (5.4). In Fig. 3 we show the resulting amplitude in the first approximation, i.e., retaining expansion terms up to the first order. These figures bear evidence of the wave damping which is solely due to the attenuation factor in (5.4).

In the second example we look for complex values of $h_0^{(1)}$, $h_0^{(2)}$, and choose material parameters for $\mathscr{B}_2$ in such a way that inequality (4.7) be satisfied. Accordingly, we no longer make the hypothesis $G_{45}^{(2)} = 0$ which was at the basis of the evaluations in Fig. 2 and take $G_{45}^{(2)} = 0.9 \times 10^{10}$ Pa, $\gamma = 1.04$ and $r = 1.77$ which correspond to an half-space $\mathscr{B}_2$ made of lithium niobate ($LiNbO_3$). In this case inequalities (4.7) are again satisfied while (4.9) does not hold. As noted in Section 4, k_1 turns out to be real and $h_0^{(q)}, h_1^{(q)}$ take complex

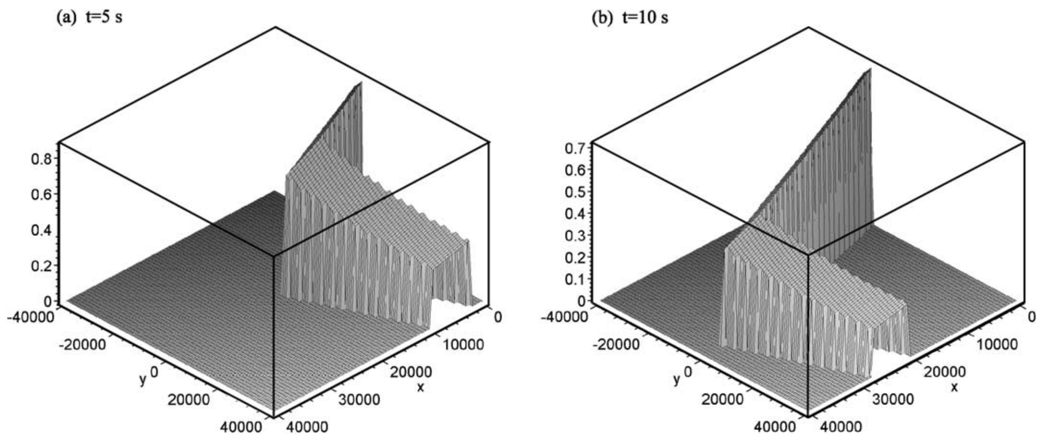

Fig. 3. The amplitude of a transient solution corresponding to a square pulse with $\tau_{\rm p} = 2s$ for the first example, in the plane (x, y) at two different times: (a) $t = 5s$ and (b) $t = 10s$.

Table 3
Wave parameters for the second example

$k_0 = -2.880 \times 10^{-4}$ s/m	$k_1 = -5.533 \times 10^{-6}$ m^{-1}
$h_0^{(1)} = (-8.904 \times 10^{-5} \pm 5.889 \times 10^{-5}{\rm i})$ s/m	$h_1^{(1)} = (-1.952 \times 10^{-5} \pm 4.972 \times 10^{-5}{\rm i})$ m^{-1}
$h_0^{(2)} = (-4.305 \times 10^{-5} \mp 5.663 \times 10^{-5}{\rm i})$ s/m	$h_1^{(2)} = (-9.437 \times 10^{-6} \mp 4.781 \times 10^{-5}{\rm i})$ m^{-1}
$c_k = 3472$ m/s	$c^{(1)} = 3317$ m/s, $c^{(2)} = 3434$ m/s

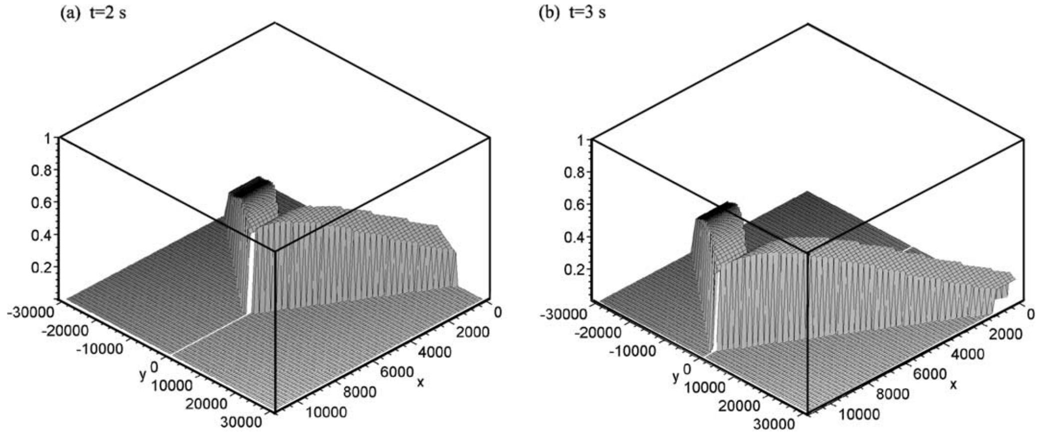

Fig. 4. The amplitude of a transient solution corresponding to a square pulse with $\tau_p = 0.5s$ for the second example, in the plane (x, y) at two different times: (a) $t = 2s$ and (b) $t = 3s$.

values in the form of conjugate pairs. The results for the corresponding wave parameters are shown in Table 3 for $\kappa = 0.1$.

As to the wave behavior, also in this case we have considered an initial square pulse in the form of Eq. (6.1). Fig. 4 shows the propagating pulse in the same approximation as in the previous example. In this case we remark that wave amplitudes vanish at large distances from $\mathscr{S}$ as $1/y$.

7. Concluding remarks

We have shown that the interface $\mathscr{S}$ between two viscoelastic anisotropic half-spaces $\mathscr{B}_1$, $\mathscr{B}_2$ can drive SH transient waves whose amplitude vanishes at large distances from $\mathscr{S}$. The solutions presented here can be viewed as a simple generalization of the stationary case since they are based on a separation of space variables which allows us to write the mechanical displacement as the convolution of two spatially independent fields. Owing to this assumption, we obtain plane wave fronts propagating in $\mathscr{B}_1$ and $\mathscr{B}_2$.

It is worth remarking that, in the case of purely elastic solids, the solutions obtained in Section 5 still exist in a simpler form. In fact, in absence of viscoelasticity, Eq. (5.4) yields

$$\mathscr{P}_a^{(q)}(x, y, t) = \delta\left(t - \frac{x}{c_k} - (-1)^q \zeta_0^{(q)} y\right), \quad q = 1, 2, \tag{7.1}$$

and, from Eq. (5.8), we obtain

$$\mathscr{P}_b^{(q)}(x, y, t) = \frac{\eta_0^{(q)}}{\pi} \frac{y}{\eta_0^{(q)^2} y^2 + t^2}, \quad q = 1, 2. \tag{7.2}$$

We note that the first class of solutions, represented by Eq. (7.1), consists of not decaying homogeneous transient waves. The second class of solutions, represented by the convolution of the propagators (7.1) and (7.2), consists of inhomogeneous transient waves whose amplitude decreases as $1/y$ at large distances from $\mathscr{S}$.

Finally, one can consider a viscoelastic/elastic interface. In this case the viscoelastic solutions (5.4) and (5.8) can be matched at the interface with the corresponding elastic solutions (7.1) and (7.2).

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